JOURNAL OF APPROXIMATION THEORY **61**, 106–117 (1990)

# Markov's Inequality and the Existence of an Extension Operator for $C^\infty$ Functions

W. PLEŚNIAK

*Department of Mathematics, Jagiellonian University, 30-059 Kraków, Poland*

*Communicated by Paul Nevai*

Received March 25, 1988; revised June 20, 1988

It is shown that if $E$ is a $C^\infty$ determining compact set in $\mathbb{R}^n$, then Markov's inequality for derivatives of polynomials holds on $E$ iff there exists a continuous linear extension operator $L$: $C^\infty(E) \to C^\infty(\mathbb{R}^n)$. Other equivalent statements (e.g., Bernstein's approximation theorem for $C^\infty$ functions, topological linear embedding of $C^\infty(E)$ into the space of rapidly decreasing sequences of real numbers) are also given. As an application, we prove that each of those properties (of the set $E$) is invariant under regular analytic mappings. © 1990 Academic Press, Inc.

## 0. INTRODUCTION

Perhaps one of the most laborious tasks in approximation theory is to compile the list of papers dealing with the classical Markov's inequality and its generalizations. A contribution to this theory has been given by W. Pawłucki and the author in [2], where it is shown that, for each polynomial $p$: $\mathbb{K}^n \to \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and each multiindex $\alpha$,

$$\|D^\alpha p\|_E \leqslant M(\deg p)^{r|\alpha|}\, \|p\|_E) \tag{0.1}$$

with positive constants $M$ and $r$ independent of $p$ and $\alpha$, whenever $E$ is a *uniformly polynomially cuspidal* (UPC) compact subset of $\mathbb{K}^n$. (Here $\|p\|_E$ stands for sup $|p|$ $(E)$.) Actually, in order that (0.1) hold it is sufficient that *Siciak's extremal function* $\Phi_E$ (see [7]), defined by the formula

$$\Phi_E(x) = \sup\{|p(x)|^{1/\deg p}: p \text{ is a polynomial on } \mathbb{C}^n \text{ of degree} \geqslant 1 \text{ with } \|p\|_E \leqslant 1\},$$

for $x \in \mathbb{C}^n$, be Hölder continuous in the sense that

$$\Phi_E(x) \leqslant 1 + M\delta^m \qquad \text{whenever } \operatorname{dist}(x, E) \leqslant \delta \leqslant 1, \tag{HCP}$$



0021-9045/90 $3.00
Copyright © 1990 by Academic Press, Inc.
All rights of reproduction in any form reserved.

with $M>0$ and $m>0$ independent of $\delta$ (see [2, Remark 3.2]). Recently Siciak (personal communication) has constructed a Cantor's type compact subset $E$ of the line-segment $[0, 1]$ that satisfies (HCP). Thus, the family of HCP compact sets in $\mathbb{R}^n$ is strictly larger than that of UPC sets. We note that by the famous Hironaka's rectilinearization theorem and Łojasiewicz's inequality, every subanalytic compact set $E$ in $\mathbb{R}^n$ with $\operatorname{int} E$ dense in $E$ is UPC [2, Corollary 6.6]. As an application of Markov's inequality it has also been proved in [2, Theorem 5.1] that a function $f: E \to \mathbb{R}$ is the restriction of a $C^\infty$ function on $\mathbb{R}^n$ iff for each $r>0$, $\lim_{k\to\infty} k^r \operatorname{dist}_E(f, P_k) = 0$ (Bernstein's theorem), $P_k$ denoting the space of (the restrictions to $E$ of) all polynomials of degree at most $k$, and

$$\operatorname{dist}_E(f, P_k) := \inf\{\|f-p\|_E : p \in P_k\}.$$

Applying both Markov's inequality and Bernstein's theorem, we have shown in [4] (see also [3]) the existence of a continuous linear operator extending $C^\infty$ functions from an HCP compact subset of $\mathbb{R}^n$ to the whole space and have constructed [5] a topological linear embedding of $C^\infty(E)$ into the Fréchet space $s$ of rapidly decreasing sequences of real numbers.

This paper completes the previous articles by W. Pawłucki and the author. Here, applying the techniques developed in [2, 4, 5], we point out that Markov's inequality, as well as Bernstein's theorem, is equivalent to the existence of a continuous linear extension operator for $C^\infty$ functions on $E$ in the case that $E$ is a compact subset of $\mathbb{R}^n$, $C^\infty$ *determining* in the following sense: For each $f \in C^\infty(\mathbb{R}^n)$, $f_{|E}=0$ implies $D^\alpha f_{|E}=0$, for all $\alpha \in \mathbb{Z}_+^n$. This seems not to be known to specialists in the field. As an application, we show that each of the equivalent properties of $E$ is invariant under regular analytic maps. We close the paper by formulating some open problems concerning sets with Markov's property.

## 1. $C^\infty$ Functions

A $C^\infty$ function on a compact subset $E$ of $\mathbb{R}^n$ is a function $f: E \to \mathbb{R}$ such that there exists a function $\tilde{f} \in C^\infty(\mathbb{R}^n)$ with $\tilde{f}_{|E} = f$. Let $C^\infty(E)$ be the space of such functions. Following Zerner [12] we introduce in $C^\infty(E)$ the seminorms $d_{-1}(f) := \|f\|_E$, $d_0(f) := \operatorname{dist}_E(f, P_0)$, and for $k=1, 2, \ldots$,

$$d_k(f) := \sup_{l \geq 1} l^k \operatorname{dist}_E(f, P_l).$$

By Jackson's theorem (see, e.g., [10]) the $d_k$'s are indeed seminorms on $C^\infty(E)$. Denote by $\tau_1$ the topology for $C^\infty(E)$ determined by the seminorms $d_k$ $(k=-1, 0, \ldots)$. In general, this topology is not complete.

Let now $\tau_2$ be another topology for $C^\infty(E)$ determined by the seminorms

$$q_{K,k}(f) := \inf\{|\tilde{f}|_K^k : \tilde{f} \in C^\infty(\mathbb{R}^n), \tilde{f}_{|E} = f\},$$

where for each compact set $K$ in $\mathbb{R}^n$ and each $k = 0, 1, \ldots,$

$$|\tilde{f}|_K^k := \max_{|\alpha| \leqslant k} \|D^\alpha \tilde{f}\|_K.$$

Then $\tau_2$ is exactly the quotient topology of the space $C^\infty(\mathbb{R}^n)/I(E)$, where $C^\infty(\mathbb{R}^n)$ is endowed with the natural topology $\tau_0$ determined by the seminorms $|\cdot|_K^k$, and $I(E) := \{f \in C^\infty(\mathbb{R}^n) : f_{|E} = 0\}$. Since $(C^\infty(\mathbb{R}^n), \tau_0)$ is complete and $I(E)$ is a closed subspace of $C^\infty(\mathbb{R}^n)$, the quotient space $C^\infty(\mathbb{R}^n)/I(E)$ is also complete, whence $(C^\infty(E), \tau_2)$ is a Fréchet space.

Suppose now $E$ is a $C^\infty$ determining compact set in $\mathbb{R}^n$. Then by Whitney's extension theorem the space $(C^\infty(E), \tau_2)$ is isomorphic to the Fréchet space of $C^\infty$ *Whitney fields* $F = (F^\alpha)$ $(\alpha \in \mathbb{Z}_+^n)$, where each $F^\alpha$ is a continuous function on $E$, endowed with the topology $\tau_3$ determined by the seminorms

$$\|F\|_E^k := |F|_E^k + \sup\{|(R_x^k F)^\alpha (y)/|x-y|^{k-|\alpha|} : x, y \in E, x \neq y, |\alpha| \leqslant k\},$$

$(k = 0, 1, \ldots)$, where

$$|F|_E^k = \sup\{|F^\alpha(x)| : x \in E, |\alpha| \leqslant k\}$$

and

$$(R_x^k F)^\alpha (y) = F^\alpha(y) - \sum_{|\beta| \leqslant k - |\alpha|} (1/\beta!) F^{\alpha+\beta}(x)(y-x)^\beta.$$

Assume, moreover, that the compact set $E$ has the $C^\infty$ *extension property*: Every $C^\infty$ function on $\operatorname{int} E$ that is uniformly continuous together with all its partial derivatives can be extended to a $C^\infty$ function on $\mathbb{R}^n$. Then the topologies $\tau_2$ and $\tau_3$ coincide with the topology $\tau_4$ determined by the seminorms

$$|f|_K^k = \sup\{|D^\alpha f(x)| : x \in \operatorname{int} E, |\alpha| \leqslant k\}.$$

(For details see [4].) Further on, we shall need the following known

LEMMA 1.1 (see, e.g., [1, I.4.2]). *There are positive constants $C_\alpha$ depending only on $\alpha \in \mathbb{Z}_+^n$ such that for each compact set $E$ in $\mathbb{R}^n$ and each $\varepsilon > 0$, one can find a $C^\infty$ function $u$ on $\mathbb{R}^n$ satisfying $0 \leqslant u \leqslant 1$ on $\mathbb{R}^n$, $u = 1$ in a neighborhood of $E$, $u(x) = 0$ if* $\operatorname{dist}(x, E) > \varepsilon$, *and for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}_+^n$, $|D^\alpha u(x)| \leqslant C_\alpha \varepsilon^{-|\alpha|}$.*

## 2. Lagrange Interpolation Polynomials

Let $\kappa : \mathbb{N} \ni j \to \kappa(j) = (\kappa_1(j), ..., \kappa_n(j)) \in \mathbb{Z}^n_+$ be a one-to-one mapping such that for each $j$, $|\kappa(j)| \leqslant |\kappa(j+1)|$. Let $m_k$ denote the number of all monomials $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ of degree at most $k$. One can easily verify that $m_k = \binom{n+k}{k}$. Set $e_j(x) := x^{\kappa(j)}$, for $j = 1, 2, ....$ The system $\{e_1, ..., e_{m_k}\}$ is a basis of the space $P_k$ of all polynomials from $\mathbb{K}^n$ to $\mathbb{K}$ of degree at most $k$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

Let $T_k = \{t_1, ..., t_k\}$ be a system of $k$ points of $\mathbb{K}^n$. Consider the *Vandermonde determinant*

$$V(T_k) = V(t_1, ..., t_k) := \det[e_j(t_i)],$$

where $i, j \in \{1, ..., k\}$. If $V(T_k) \neq 0$, we define

$$L^{(j)}(x, T_k) := V(t_1, ..., t_{j-1}, x, t_{j+1}, ..., t_k)/V(T_k).$$

Since $L^{(j)}(t_i, T_k) = \delta_{ij}$ (Kronecker's symbol), we get the following *Lagrange interpolation formula* (cf. [8, Lemma 2.1]):

(LIF) If $p \in P_k$ and $T_{m_k}$ is a system of $m_k$ points of $\mathbb{K}^n$ such that $V(T_{m_k}) \neq 0$, then

$$p(x) = \sum_{j=1}^{m_k} p(t_j) L^{(j)}(x, T_{m_k}), \qquad \text{for} \quad x \in \mathbb{K}^n.$$

Let $E$ be a *unisolvent* compact set in $\mathbb{K}^n$. That means that for each polynomial $p$, $p = 0$ on $E$ implies $p = 0$ in $\mathbb{K}^n$. A system $T_k$ of $k$ points $\{t_1, ..., t_k\}$ of $E$ is called a *Fekete–Leja system of extremal points* of $E$ of order $k$, if $|V(T_k)| \geqslant |V(S_k)|$ for all systems $S_k = \{s_1, ..., s_k\} \subset E$. Since $E$ is unisolvent, we have $V(T_k) \neq 0$ (see [8, Proposition 4.3]), and then

$$|L^{(j)}(x, T_k)| \leqslant 1 \text{ on } E, \qquad \text{for} \quad j = 1, ..., k. \tag{2.1}$$

If $f: E \to \mathbb{K}$ is a function on $E$, for any system $T_{m_k}$ of extremal points of $E$ of order $m_k$, we set

$$L_k f(x) = \sum_{j=1}^{m_k} f(t_j)\, L^{(j)}(x, T_{m_k}). \tag{2.2}$$

$L_k f$ is called the *Lagrange interpolation polynomial* of $f$ of order $k$. Suppose $f$ is continuous and $p_k$ is a metric projection of $f$ onto $P_k$. Then by LIF, (2.1), and (2.2), we get for $k \geqslant 2$

$$\begin{aligned} \|f - L_k f\|_E &\leqslant \|f - p_k\|_E + \|L_k f - L_k(p_k)\|_E \\ &\leqslant (m_k + 1)\, \|f - p_k\|_E \leqslant 4k^n \operatorname{dist}_E(f, P_k). \end{aligned} \tag{2.3}$$

Due to this inequality, in problems connected with polynomial approximation of $C^\infty$ functions one can well replace the metric projection $f \to \operatorname{dist}_E(f, P_k)$ by the linear projection $f \to L_k f$.

## 3. The main Result

Let $E$ be a unisolvent compact set in $\mathbb{R}^n$ and $f: E \to \mathbb{R}$. Fix a point $t_0 \in E$ and set $L_0 f(x) \equiv f(t_0)$. For each $k = 1, 2, \ldots$, let $T_{m_k} = \{t_1^k, \ldots, t_{m_k}^k\} \subset E$ be a Fekete–Leja system of extremal points of $E$ of order $m_k$ and let $L_k f$ denote the Lagrange interpolation polynomial of $f$ of degree $k$ with nodes in $T_{m_k}$. Then we set $\varphi_1(f) = f(t_0)$, and for $M_k < j \leqslant M_{k+1}$, where $M_k := m_0 + \cdots + m_k$ for $k = 0, 1, \ldots$, we define

$$\varphi_j(f) = (L_{k+1} f - L_k f)(t_{j-M_k}^{k+1}).$$

In [5] we have proved

Proposition 3.1. *For any unisolvent compact subset $E$ of $\mathbb{R}^n$, the assignment $\varphi: f \to \{\varphi_j(f)\}_{j=1}^\infty$ determines a topological linear embedding of the space $(C^\infty(E), \tau_1)$ into the Fréchet space $\mathfrak{s}$ of all rapidly decreasing sequences of real numbers furnished with the norms $|x|_k := \sup_j j^k |x_j|$, $k = 0, 1, \ldots$.*

In order to make this article self-contained we shall repeat the argument of [5]. It is clear that $\varphi$ is linear. By LIF (Section 2) and (2.3), $\varphi$ is injective. To prove that $\varphi(f) \in \mathfrak{s}$, observe that by (2.3), for each $r = 0, 1, \ldots$, and $M_k < j \leqslant M_{k+1}$, $k = 1, 2, \ldots$, we get

$$\begin{aligned} |j^r \varphi_j(f)| &\leqslant M_{k+1}^r \|L_{k+1} f - L_k f\|_E \\ &\leqslant C k^{(n+1)r+n} \operatorname{dist}_E(f, P_k) \leqslant C d_{(n+1)r+n}(f), \end{aligned}$$

where $C$ is a constant depending only on $n$ and $r$. If $1 \leqslant j \leqslant M_1$, we also have $|\varphi_j(f| \leqslant 2m_1 d_{-1}(f)$. Hence, in particular, $\varphi$ is continuous. To prove that $\varphi$ is a homeomorphism (onto $\varphi(C^\infty(E))$ it suffices to show that for each $r = -1, 0, \ldots$ there are a positive integer $s$ and a constant $M > 0$ such that $d_r(f) \leqslant M|\varphi(f)|_s$. To this end, write $q_0 := L_0 f$ and $q_k := L_k f - L_{k-1} f$, if $k > 0$. Since $f = \sum_{k=0}^\infty q_k$ on $E$, for $r > 0$ we have

$$\begin{aligned} l^r \operatorname{dist}_E(f, P_l) \leqslant l^r \sum_{k=l+1}^\infty \|q\|_E \leqslant \sum_{k=l+1}^\infty k^r \|q_k\|_E \\ \leqslant (\pi^2/6) \sup_{k \geqslant 1} k^{r+2} \|q_k\|_E. \end{aligned}$$

On the other hand, by LIF we get

$$\|q_k\|_E = \sup_{x \in E} \sum_{i=1}^{m_k} |q_k(t_i^k) L^{(i)}(x, T_{m_k})| \leqslant m_k \max_{1 \leqslant i \leqslant m_k} |q_k(t_i^k)|.$$

Hence, since for each $k \geqslant 1$ we have $k \leqslant M_{k-1}$, it follows that

$$\begin{aligned} d_r(f) &\leqslant M \sup_{k \geqslant 1} k^{r+n+2} \max\{|\varphi_j(f)| : M_{k-1} < j \leqslant M_k\} \\ &\leqslant M\, |\varphi(f)|_{r+n+2} \end{aligned}$$

with $M$ depending only on $n$. If $r = 0$, we also have

$$d_0(f) \leqslant \|f - L_0 f\|_E \leqslant \sum_{k=1}^{\infty} \|q_k\|_E \leqslant (\pi^2/3 + 1)\, |\varphi(f)|_{n+2}.$$

This completes the proof of the proposition.

*Remark* 3.2. In general, $\varphi$ is not a surjection on $\mathfrak{s}$. The reason is that for $k \neq l$, the set $T_{m_k}$ may meet the set $T_{m_l}$.

Now, our main result reads as follows.

THEOREM 3.3. *If $E$ is a $C^\infty$ determining compact subset of $\mathbb{R}^n$ then the following statements are equivalent*:

(i) (Markov's Inequality) *There exist positive constants $M$ and $r$ such that for each polynomial $p$ and each $\alpha \in \mathbb{Z}_+^n$,*

$$\|D^\alpha p\|_E \leqslant M(\deg p)^{r|\alpha|}\, \|p\|_E.$$

(ii) *There exist positive constants $M$ and $r$ such that for every polynomial $p$ of degree at most $k$, $k = 1, 2, \ldots$,*

$$|p(x)| \leqslant M\, \|p\|_E \qquad \textit{if} \quad x \in \tilde{E}_k := \{x \in \mathbb{C}^n : \operatorname{dist}(x, E) \leqslant 1/k^r\}.$$

(ii′) *There exist positive constants $M$ and $r$ such that for every polynomial $p$ of degree at most $k$, $k = 1, 2, \ldots$,*

$$|p(x)| \leqslant M\, \|p\|_E \qquad \textit{if} \quad x \in E_k := \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) \leqslant 1/k^r\}.$$

(iii) (Bernstein's Theorem) *For every function $f: E \to \mathbb{R}$, if the sequence $\{\operatorname{dist}_E(f, P_l)\}$ is rapidly decreasing, then there is a $C^\infty$ function $\tilde{f}$ on $\mathbb{R}^n$ such that $\tilde{f}|_E = f$.*

(iv) *The space $(C^\infty(E), \tau_1)$ is complete.*

(v) *The topologies $\tau_1$ and $\tau_2$ for $C^\infty(E)$ coincide.*

(vi) *The mapping* $\varphi$ *of Proposition* 3.1 *is a linear homeomorphism of* $(C^\infty(E), \tau_2)$ *onto its image in* $\mathfrak{s}$.

(vii) *There exists a continuous linear operator*

$$L\colon (C^\infty(E), \tau_1) \to (C^\infty(\mathbb{R}^n), \tau_0)$$

*such that* $Lf_{|E} = f$ *for each* $f \in C^\infty(E)$.

The proof consists of four steps. 1° (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (ii′). Assume (i). Let $p$ be a polynomial of degree at most $k$. For each $x \in \mathbb{C}^n$, there is $a \in E$ such that $\delta := \operatorname{dist}(x, E) = |x-a|$. By Taylor's formula we have

$$p(x) = \sum_{|\alpha| \leqslant k} (D^\alpha p(a)/\alpha!)(x-a)^\alpha.$$

Hence by (i), for $\delta = |x-a|$,

$$|p(x)| \leqslant M \sum_{|\alpha| \leqslant k} (k^r \delta)^{|\alpha|} \, \|p\|_E/\alpha! = M \, \|p\|_E \sum_{l=0}^{k} (nk^r\delta)^l/l!.$$

By putting $\delta \leqslant 1/k^r$ we get

$$|p(x)| \leqslant Me^n \, \|p\|_E \qquad \text{if} \quad \operatorname{dist}(x, E) = 1/k^r.$$

(ii) $\Rightarrow$ (ii′) Trivial. (ii′) $\Rightarrow$ (i) For each $a \in E$, $I_k(a) := \{x \in \mathbb{R}^n : |x_j - a_j| \leqslant 1/n^{1/2}k^r, j = 1, \ldots, n\} \subset E_k$. Hence, by the classical Markov's inequality for a cube,

$$\begin{aligned} |D^\alpha p(a)| &\leqslant [k^2/(1/n^{1/2}k^r)]^{|\alpha|} \, \|p\|_{I_k(a)} \\ &\leqslant Mn^{|\alpha|/2} \cdot k^{(r+2)|\alpha|} \, \|p\|_E \leqslant M_1 k^{(r+2+s)|\alpha|} \, \|p\|_E \end{aligned}$$

for each $\alpha \in \mathbb{Z}^n_+$. (Here the constant $M_1$ is determined by the equivalence of norms on the space $P_1$ of polynomials of degree at most 1, and $s$ is chosen so that $n^{1/2} \leqslant 2^s$.)

2° (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). By step 1°, (i) implies (ii) and then we can follow the argument of the proof of Bernstein's theorem in [2]: Suppose $f\colon E \to \mathbb{R}$ is such a function that for each $s > 0$, $\lim_{k\to\infty} k^s \, \|f - p_k\|_E = 0$, $p_k$ being a metric projection of $f$ onto $P_k$ $(k = 0, 1, \ldots)$. Set $\varepsilon_k = 1/k^r$ with (an integer) $r$ determined by both (i) and (ii), and for $k = 1, 2, \ldots$, take a function $u_k$ of Lemma 1.1 corresponding to $\varepsilon_k$. Then the assignement

$$\tilde{f} := p_0 + \sum_{k=1}^{\infty} u_k q_k,$$

where $q_k := p_k - p_{k-1}$, $k = 1, 2, \ldots$, determines a $C^\infty$ function $\tilde{f}$ on $\mathbb{R}^n$ such that $\tilde{f}_{|E}=f$. For, if $E_k=\{x\in\mathbb{R}^n : \operatorname{dist}(x, E)\leqslant\varepsilon_k\}$ and $\alpha\in\mathbb{Z}^n_+$, by (i) and (ii) we get

$$\begin{aligned}\sup_{\mathbb{R}^n}|D^\alpha(u_kq_k)| &\leqslant \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\sup_{E_k}|D^\beta u_k D^{\alpha-\beta}q_k|\\ &\leqslant M\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}C_\beta k^{r|\beta|}\,\|D^{\alpha-\beta}q_k\|_E\\ &\leqslant M_1k^{r|\alpha|}\|q_k\|_E\leqslant M_2k^{-2}d_{r|\alpha|+2}(f)\end{aligned}$$

with a constant $M_2$ independent of $k$.

Assume now (iii). Then (iv) follows by continuity of the map $C(E)\ni f\to\operatorname{dist}_E(f, P_k)\in\mathbb{R}$ $(k=0, 1, ...)$, where $C(E)$ is the Banach space of all continuous functions on $E$ with the supremum norm. Let now $I$ be a compact cube in $\mathbb{R}^n$ containing $E$ in its interior. By Jackson's theorem (see. e.g., [10]), for every $k$ there is a constant $C_k>0$ such that for each $f\in C^\infty(E)$,

$$d_k(f)\leqslant C_kq_{I,k+1}(f).$$

Hence, if $(C^\infty(E), \tau_1)$ is complete, by Banach's theorem the topologies $\tau_1$ and $\tau_2$ are equal, and we get (v). (We recall that for any $C^\infty$ determining compact set $E$ in $\mathbb{R}^n$ the topology $\tau_2$ is equal to the topology $\tau_3$. If, moreover, $E$ has the $C^\infty$ extension property, then both $\tau_2$ and $\tau_3$ are equal to the topology $\tau_4$.)

If (v) holds, there are a positive constant $M$ and an integer $r\geqslant -1$ such that for each $f\in C^\infty(E)$, we have $q_{E,1}(f)\leqslant Md_r(f)$. Since $d_{-1}(f)=\|f\|_E$ and $d_0(f\leqslant\|f\|_E$, it must be $r\geqslant 1$. (Otherwise, consider the functions $x_j^m$, $m=1, 2, ....$) In particular, if $f$ is a polynomial of degree at most $k$, we get

$$\|(\partial f/\partial x_j)\|_E\leqslant M\sup_{1\leqslant l\leqslant k} l^r\operatorname{dist}_E(f, P_l)\leqslant Mk^r\,\|f\|_E,$$

for $j=1, 2, ..., n$, which implies (i). (We needed the assumption that $E$ is $C^\infty$ determining.)

3° (v)$\Leftrightarrow$(vi). If the topologies $\tau_1$ and $\tau_2$ are equal then (vi) follows by Proposition 3.1, since every $C^\infty$ determining compact set $E$ is obviously unisolvent. Conversely, (vi) implies that the identity map $I$: $(C^\infty(E), \tau_1)\to(C^\infty(E), \tau_2)$ is continuous. Hence by Jackson's theorem, $I$ is a linear homeomorphism.

4° (i)$\Leftrightarrow$(vii). The existence of an extension operator $L$: $(C^\infty(E), \tau_1)\to(C^\infty(\mathbb{R}^n), \tau_0)$ has been proved in [4] (see also [3]) under the assumption that $E$ is HCP. An inspection of the proof of that result permits us to repeat the argument under the hypothesis (i). For, let $u_k$ be the functions of the proof of implication (i) ⇒ (iii) (step 2°) and let $L_k f$ be the Lagrange interpolation polynomials of $f \in C^\infty(E)$ corresponding to the Fekete-Leja extremal points of $E$ (Section 2). Then the operator

$$Lf = u_1 L_1 f + \sum_{k=1}^{\infty} u_k(L_{k+1} f - L_k f)$$

is obviously linear and $Lf = f$ on $E$. Moreover, by (i), (ii), and (2.3), for each $\alpha \in \mathbb{Z}^n_+$ we have

$$\begin{aligned}
&\sup_{\mathbb{R}^n} |D^\alpha u_k(L_{k+1} f - L_k f)| \\
&\leqslant \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \sup_{E_k} |D^\beta u_k D^{\alpha-\beta}(L_{k+1} f - L_k f)| \\
&\leqslant \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} M C_\beta k^{r|\beta|} \, \|D^{\alpha-\beta}(L_{k+1} - L_k f)\|_E \\
&\leqslant M_\alpha k^{r|\alpha|} \, \|L_{k+1} f - L_k f\|_E \\
&M'_\alpha k^{r|\alpha|+n} \operatorname{dist}_E(f, P_k) \leqslant M'_\alpha k^{-2} d_{r|\alpha|+n+2}(f).
\end{aligned}$$

Thus, $L$ is a continuous mapping from $(C^\infty(E), \tau_1)$ to $(C^\infty(\mathbb{R}^n), \tau_0)$. It remains to prove that (vii) implies (i), and this goes on the same lines as the proof of implication (v) ⇒ (i).

*Remark* 3.4. The observation that (vii) implies (i) is owed to Siciak (personal communication). The equivalence (vi) ⇔ (vii) is related to a result of Tidten [9].

*Remark* 3.5. Some of the implications of Theorem 3.3 do not require the assumption that $E$ is $C^\infty$ determining. In particular, equivalences (i) ⇔ (ii) ⇔ (ii′) hold for any compact subset $E$ of $\mathbb{R}^n$.

On the other hand, if $E$ satisfies (i) then $E$ must be $C^\infty$ determining. For let $I$ be a compact cube in $\mathbb{R}^n$ containing $E$ in its interior. Take $f \in C^\infty(\mathbb{R}^n)$ with $f_{|E} = 0$. For each $k$, let $p_k$ be a polynomial of degree at most $k$ such that $\varepsilon_k := \operatorname{dist}_I(f, P_k) = \|f - p_k\|_I$. By Jackson's theorem $(\varepsilon_k)$ is rapidly decreasing, and by Markov's inequality, for each $\alpha \in \mathbb{Z}^n_+$ we get

$$D^\alpha f = D^\alpha p_1 + \sum_{k=1}^{\infty} D^\alpha(p_{k+1} - p_k) \qquad \text{on } I.$$

Thus, again by Markov's inequality (on $E$), if $x \in E$, then $D^\alpha f(x) = \lim_{k\to\infty} D^\alpha p_k(x) = 0$.

## 4. Applications

Theorem 3.3 unites two apparently different problems: Markov's inequality and the existence of a continuous linear extension of $C^\infty$ functions. This brings some advantages: e.g., we now can easily explain the phenomenon that there is no continuous linear extension operator $L: (C^\infty(E), \tau_1) \to C^\infty(\mathbb{R}^n)$ in the case that $E = \{(x, y) \in \mathbb{R}^2 : 0 < y \leqslant \exp(-1/x), 0 < x \leqslant 1\} \cup \{(0, 0)\}$. Due to Theorem 3.3, it suffices to show that $E$ does not satisfy (i), which is evident if we consider the polynomials $p_k(x, y) = y(1-x)^k$ for $k = 1, 2, ...$ (example of Zerner [12]). (Cf. also [9].)

By [2, Theorem 3.1] and [4, Proposition 1.1], the class of UPC compact sets is a subclass of sets satisfying (i) of Theorem 3.3 that is stable with respect to the diffeomorphisms. For the whole class of (nonpluripolar) compact sets with property (i), we can prove a more modest

Proposition 4.1. *Suppose $E$ is a nonpluripolar compact set in $\mathbb{R}^n$. (That means that there is no plurisubharmonic function $u$ on $\mathbb{C}^n$, $u(z) \not\equiv -\infty$, such that $E \subset \{u = -\infty\}$.) Let $h$ be an analytic mapping defined in an open neighborhood $U$ of $E$, with values in $\mathbb{R}^n$, such that for each $x \in E$.* $\det h'(x) \neq 0$. *If then $E$ satisfies* (i) *of Theorem* 3.3, *it satisfies also each of the requirements* (i)–(vii), *and so does the set* $h(E)$.

*Proof.* By Theorem 3.3 and Remark 3.5, it suffices to show that if $E$ satisfies (ii′) so does the set $h(E)$. In order to do this take $b \in h(E)$ and choose $a \in h^{-1}(b) \cap E$. By the assumptions on $h$, there exist positive constants $L$ and $L_1$ such that if $0 < \delta \leqslant \delta_0 := \inf\{|\det h'(x)| : x \in E\}$, then

$$B(b, L\delta) \subset h(B(a, L_1\delta)) \tag{4.1}$$

(see, e.g., [11, Chap. I, Prop. 5.1]). Choose an integer $k_0 > 1$ so that $1/k_0^r \leqslant \delta_0$ and

$$F := \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) \leqslant L_1/k_0^r\} \subset U.$$

Let $\tilde{h}$ be the complexification of $h$ defined in an open set $\tilde{U}$ in $\mathbb{C}^n$ containing $U$. We may assume that $\tilde{h}$ is bounded on $\tilde{U}$. Since every compact set in $\mathbb{R}^n$ is polynomially convex, by a "uniform version" of the Bernstein–Walsh theorem (see [6, Lemma 2.1]) there exist constants $M_1 > 0$ and $a \in (0, 1)$ such that for each $p \in P_k$ $(k \geqslant k_0)$

$$\operatorname{dist}_F(p \circ h, P_l) \leqslant M_1 \|\tilde{p} \circ \tilde{h}\|_{\tilde{U}} \cdot a^l, \qquad l = 1, 2, ..., \tag{4.2}$$

$\tilde{p}$ denoting the complexification of $p$. By hypothesis, $E$ is nonpluripolar.

and by [6, Lemma 2.5] so is the set $h(E)$. Hence the extremal function $\Phi_{h(E)}$ is locally bounded in $\mathbb{C}^n$ (see [8]), and by the definition of $\Phi$,

$$\|\tilde{p} \circ \tilde{h}\|_{\tilde{U}} = \|\tilde{p}\|_{\tilde{h}(\tilde{U})} \leqslant \|p\|_{h(E)} A^k, \tag{4.3}$$

where $A := \sup\{\Phi_{h(E)}(y) : y \in \tilde{h}(\tilde{U})\} < +\infty$. For each $l$, let $q_l$ be the polynomial (depending on $p$) of degree at most $l$ such that

$$\operatorname{dist}_F(p \circ h, P_l) = \|p \circ h - q_l\|_F.$$

If we put $l = dk$, where $d$ is an integer such that $Aa^d \leqslant 1$, then by (4.2) and (4.3)

$$\|p \circ h - q_l\|_F \leqslant M_1 \|p \circ h\|_E$$

and

$$\|q_l\|_E \leqslant \|p \circ h - q_l\|_F + \|p \circ h\|_E \leqslant (M_1 + 1) \|p \circ h\|_E.$$

Moreover, by (4.1), for each $k \geqslant k_0$ and $s \geqslant r$ we get

$$\begin{aligned} \|p\|_{B(b, L/k^s)} &\leqslant \|p \circ h\|_{B(a, L_1/k^s)} \\ &\leqslant \|p \circ h - q_l\|_F + \|q_l\|_{B(a, L_1/k^s)} \\ &= M_1 \|p \circ h\|_E + \|q_l\|_{B(a, L_1/k^s)}. \end{aligned}$$

If now $s - r \geqslant (\log L_1 d^r)/\log k_0$, then by (ii′) for $E$ we have

$$\|q_l\|_{B(a, L_1/k^s)} \leqslant M \|q_l\|_E \leqslant M(M_1 + 1) \|p \circ h\|_E$$

and choosing $t > s$ such that $t - s \geqslant -\log L/\log k_0$ gives

$$\|p\|_{B(b, 1/k^t)} \leqslant \|p\|_{B(b, L/k^s)} \leqslant M_2 \|p\|_{h(E)} \tag{4.4}$$

with constants $M_2$ and $t$ that are independent of $p$, $k$, and $b$. Since $h(E)$ is nonpluripolar, the inequalities (4.4) extend easily to the case where $1 \leqslant k \leqslant k_0$. The proof is concluded.

*Open Problems*

1° Does (i) of Theorem 3.3 imply that the extremal function $\Phi_E$ is continuous in $\mathbb{C}^n$? In particular, does (i) imply that $E$ is nonpluripolar?

2° Construct a compact set $E$ in $\mathbb{R}^n$ that satisfies (i) and does not have HCP.

3° Does the Cantor ternary set in $\mathbb{R}$ satisfy (i)?

## ACKNOWLEDGMENT

We thank W. Pawłucki for helpful conversations on this material.

## REFERENCES

1. B. MALGRANGE, "Ideals of differentiable functions," Oxford Univ. Press, London/New York, 1966.
2. W. PAWŁUCKI AND W. PLEŚNIAK, Markov's inequality and $C^\infty$ functions on sets with polynomial cusps, *Math. Ann.* **275** (1986), 467–480.
3. W. PAWŁUCKI AND W. PLEŚNIAK, Prolongement de fonctions $C^\infty$, *C. R. Acad. Sci. Paris Sér. I* **304** (1987), 167–168.
4. W. PAWŁUCKI AND W. PLEŚNIAK, Extension of $C^\infty$ functions from sets with polynomial cusps, *Studia Math.* **88** (1988), 279–287.
5. W. PAWŁUCKI AND W. PLEŚNIAK, Approximation and extension of $C^\infty$ functions defined on compact subsets of $\mathbb{C}^n$, *in* "Deformations of Mathematical Structures," 283–295 (J. Ławrynowicz, Ed.), Kluwer Academic Publishers, 1989.
6. W. PLEŚNIAK, Invariance of the *L*-regularity of compact sets in $C^N$ under holomorphic mappings, *Trans. Amer. Math. Soc.* **246** (1978), 373–383.
7. J. SICIAK, On some extremal functions and their applications in the theory of analytic functions of several complex variables, *Trans. Amer. Math. Soc.* **105** (1962), 322–357.
8. J. SICIAK, Extremal plurisubharmonic functions in $\mathbb{C}^n$, *Ann. Polon. Math.* **39** (1981), 175–211.
9. M. TIDTEN, Fortsetzungen von $C^\infty$-Funktionen, welche auf einer abgeschlossenen Menge in $\mathbb{R}^n$ definiert sind, *Manuscripta Math.* **27** (1979), 291–312.
10. A. F. TIMAN, "Theory of Approximation of Functions of a Real Variable," Pergamon, Oxford, 1963.
11. J. CL. TOUGERON, "Idéaux de fonctions différentiables," Springer, Berlin/New York, 1972.
12. M. ZERNER, Développement en séries de polynômes orthonormaux des fonctions indéfiniment différentiables, *C. R. Acad. Sci. Paris Sér. I* **268** (1969), 218–220.